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More about a bosonic string with a topological term

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Abstract. We study a model of a bosonic string in a specific antisymmetric background. It is shown that the action in the conformal gauge possesses the same residual w_∞ symmetry as that for the ordinary bosonic string. Static solutions to the classical equations of motion are found for arbitrary space–time dimension and for arbitrary orientation of the string, and the static potential in the semiclassical (quadratic) approximation is derived. It is shown that the quasi-static picture breaks down beyond some critical distance which depends on the orientation of the string.

1. Introduction

Strings in non-trivial backgrounds have been studied extensively in the last decade. The background fields are vacuum expectation values for the massless string excitations [1, 2], which in a compatible theory have to satisfy the corresponding equations of motion. These equations turn out to be equivalent [3, 4] to consistency conditions which guarantee the conformal invariance of the (first quantized) string theory.

Symmetries of superstrings in highly non-trivial backgrounds play an important role in the attempt to derive a realistic low-energy effective action (see, e.g., [5]).

In the present paper we consider a bosonic string model [6] in space–time with flat (Euclidean) metric and constant torsion. The topological term in the action [6] contains the torsion potential (an antisymmetric tensor) which is chosen to be invariant (up to gauge transformations) with respect to space–time rotations when the space–time is three dimensional. In the general case ($d > 3$) the topological term breaks down the rotational invariance and picks up a three-dimensional subspace. It has been proved [7] that such a model is equivalent to a $WO(p, q)$ gravity if the number of the space–time dimensions is $p + q = 3, 4$. The model can be considered as a generalization of the Narain–Sarmadi–Witten model [8] (see also [9]). We will show that in a conformal gauge the model admits (at the classical level) a residual infinite-dimensional w_∞ symmetry. There is, however, a conformal anomaly [10] in the quantized theory. Let us note that the critical dimension ($d = 26$) is not affected by the inclusion of the topological term in the action. Indeed, the

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unrenormalized effective action [4] contains some additional divergences in the presence of the topological term. They have to be compensated, however, by renormalization of the target-space metric and the torsion potential. As a result the topological term contributes (at the one-loop level) to the beta functions [3, 4] corresponding to these background fields [11] but does not contribute to the dilaton beta function. We also note that all three beta functions cannot vanish simultaneously in the model under consideration for any number of space–time dimensions—the particular background does not match the conditions of conformal invariance [3, 4]. It does not, therefore, correspond to a true string vacuum—the ground state is expected to be unstable (at least for a closed string).

We derive an expression for the one-loop correction to the ground-state energy (static potential) for an open string with fixed ends in space–time of arbitrary dimension D. Static solutions are found for arbitrary string orientation with respect to the three-dimensional subspace, distinguished by the topological term in the action, and the effect of quantum fluctuations is evaluated. We also discuss the (in)stability of the closed static solutions.

The paper is organized as follows. In section 2 symmetries of the model are discussed. The static solutions to the classical equations of motion are studied in section 3. In section 4 an expression is derived for the static potential. The (in)stability of the closed static solutions is discussed in section 5 and some concluding remarks are given in section 6.

2. A string with a topological term

Let us consider a string action written as a sum

$$S = S_{\rm NG} + S_{\rm WZ} \tag{2.1}$$

of the Nambu–Goto action for a bosonic string in Euclidean space–time E^D of dimension D

$$S_{\rm NG} = \kappa \int {\rm d}^2\xi\ \sqrt{g} \tag{2.2}$$

and a trivial Wess–Zumino term

$$S_{\rm WZ} = -\frac{1}{2} \int {\rm d}^2\xi\ \epsilon^{ab} B_{\mu\nu}(X(\xi))\partial_a X^\mu \partial_b X^\nu. \tag{2.3}$$

Here X^μ ($\mu = 0, 1, ..., D-1$) are coordinates for E^D, ξ^a ($a = 0, 1$) are coordinates for the world sheet, g is the determinant of the induced metric tensor:

$$g = \det g_{ab} \qquad g_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\mu}{\partial \xi^b} \tag{2.4}$$

and B is an antisymmetric tensor field. The term $S_{\rm WZ}$ is similar to the interaction term for a particle in an electromagnetic field [12, 13]. The string dynamics is invariant under gauge transformations

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \tag{2.5}$$

where Λ_μ are the coefficients of an arbitrary 1-form on E^D. The gauge invariant quantity

$$H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu} + \partial_\mu B_{\nu\lambda} \tag{2.6}$$

is analogous to the Maxwell tensor.

As is well known, $S_{\rm NG}$ is invariant under the isometries of E^D. In general $S_{\rm WZ}$ breaks down this symmetry. An antisymmetric tensor field $H^{\lambda\mu\nu}$ respecting the isometries of E^D exists for $D = 3$. The corresponding string model for $D = 3$ has already been discussed [6, 14]. In the general case $D > 3$, which we are going to consider, any non-zero tensor field $H^{\lambda\mu\nu}$ breaks the rotational invariance. We choose a background field B which, in our reference frame, is given by

$$\begin{aligned} B_{\mu\nu} &= \tilde{\kappa}\sum_{\alpha=0}^{2}\epsilon_{\alpha\mu\nu}X^{\alpha} \qquad \mu,\nu = 0,1,2 \\ B_{\mu\nu} &= 0 \qquad \mu \ \text{ or } \ \nu > 2 \end{aligned} \tag{2.7}$$

ϵ being the Levi-Civita symbol, $\epsilon_{012} = 1$. The corresponding field $H^{\lambda\mu\nu}$ (which is a tensor field in the D-dimensional space–time E^D) is invariant under rotations belonging to a subgroup $SO(3)\otimes SO(D-3)$ of $SO(D)$. This field is invariant under translations of E^D as well. The action of a string in the background (2.7) is written as

$$S = \int \mathrm{d}^2\xi \ \left(\kappa\sqrt{g} - \frac{\tilde{\kappa}}{2}\epsilon^{ab}\epsilon_{\alpha\beta\gamma}X^{\alpha}\partial_a X^{\beta}\partial_b X^{\gamma}\right). \tag{2.8}$$

It is invariant (generally, up to a surface term) with respect to the same transformations.

Calculating the canonical momenta one finds the constraints [6]

$$\begin{aligned} \Phi &= PX_{,1} \approx 0 \\ \Phi_{\perp} &= P^2 - \kappa^2 X_{,1}^2 + 2\tilde{\kappa}\epsilon_{\alpha\beta\gamma}P^{\alpha}X^{\beta}X_{,1}^{\gamma} + \tilde{\kappa}^2(\tilde{X}^2\tilde{X}_{,1}^2 - (\tilde{X}\tilde{X}_{,1})^2) \approx 0. \end{aligned} \tag{2.9}$$

Here

$$\begin{aligned} P_{\alpha} &= \kappa\sqrt{g}g^{0b}X_{\alpha,b} - \tilde{\kappa}\epsilon_{\alpha\beta\gamma}X^{\beta}X_{,1}^{\gamma} \qquad \alpha = 0,1,2 \\ P_j &= \kappa\sqrt{g}g^{0b}X_{j,b} \qquad j = 3,\ldots,D-1 \end{aligned} \tag{2.10}$$

are the canonical momenta conjugate to X^{μ} and the notation $XY = \sum_{\mu=0}^{D-1}X^{\mu}Y^{\mu}$, $\tilde{X}\tilde{Y} = \sum_{\alpha=0}^{2}X^{\alpha}Y^{\alpha}$ is used. Notice that although the first three components of the canonical momentum differ by the corresponding momentum in standard bosonic string theory, the constraints (2.9) satisfy the same Virasoro algebra. We will show that the Virasoro algebra also appears as a Lie algebra of symmetry transformations of the Lagrangian in the conformal gauge. Moreover, the action admits an extended higher spin w-symmetry than that for the ordinary string. Another property of the ordinary string theory which is preserved here is that the canonical Hamiltonian vanishes identically.

We turn to the residual symmetry of the action (2.8) in a conformal gauge:

$$S = \int \mathrm{d}^2\xi \ (\kappa\sqrt{g} + \tilde{\kappa}\epsilon_{\alpha\beta\gamma}X^{\alpha}X_{,0}^{\beta}X_{,1}^{\gamma}). \tag{2.11}$$

We recall that the first term in this action admits w_{∞}-symmetry. We will show that the Wess–Zumino term also allows this symmetry. Indeed, let us consider the following transformation law [15]:

$$\delta_{(k)}^{n}X^{\mu} = k(\xi_+)d^{\mu\rho_1\ldots\rho_{n+1}}\partial_+X^{\rho_1}\cdots\partial_+X^{\rho_{n+1}} \qquad n = 0,1,2,\ldots \tag{2.12}$$

where $k(\xi_+)$ is an arbitrary function of one of the light-cone variables $\xi_{\pm} = \frac{1}{2}(\xi_0 \pm \xi_1)$ and d is a symmetric tensor. It is easy to check the symmetry of the action (2.11) under transformations (2.12) up to a surface term. The transformations satisfy the w_{∞}-algebra which reads

$$\left[\delta_{(k)}^{m},\delta_{(h)}^{n}\right] = \delta_{((m+1)k\partial h-(m+1)h\partial k)}^{m+n}. \tag{2.13}$$

The conserved currents corresponding to the transformations (2.12) read

$$V^n = \frac{1}{2+n}d_{\mu_1\ldots\mu_{n+2}}X_{,+}^{\mu_1}\cdots X_{,+}^{\mu_{n+2}}. \tag{2.14}$$

In the case $n = 0$, $d^{\mu\nu} = \delta^{\mu\nu}$ the first current is written as

$$V^0 = T_{++} = \tfrac{1}{2} X^{\mu}_{,+} X^{\mu}_{,+} \tag{2.15}$$

i.e. it coincides with the $(++)$-component of the stress–energy tensor being the generator of the conformal transformations. Notice that the conserved quantities (2.14) satisfy the current algebra which concurs with the Lie algebra of the infinitesimal transformations (2.13). Taking into account that the Weyl symmetry is also preserved for the action (2.8), it can be concluded that this action possesses the same symmetries as the ordinary bosonic string action.

3. Static solutions

We are going to describe the static solutions to the classical equations of motion for a string with action (2.11). These equations are written as

$$\kappa \partial_a(\sqrt{g} g^{ab} \partial_b X_\alpha) - 3\tilde{\kappa} \epsilon_{\alpha\beta\gamma} X^{\beta}_{,0} X^{\gamma}_{,1} = 0 \qquad \alpha = 0, 1, 2 \tag{3.1a}$$

$$\kappa \partial_a(\sqrt{g} g^{ab} \partial_b X_j) = 0 \qquad j = 3, \ldots, D-1. \tag{3.1b}$$

We note that for any right-moving X^α ($\partial_+ X^\alpha = 0$) or for a left-moving one ($\partial_- X^\alpha = 0$) the second term on the RHS of (3.1a) vanishes; hence they satisfy both equations (3.1). However, a linear combination of right- and left-moving functions would not, in general, be a solution to (3.1a), in contrast to the cases of ordinary bosonic string or bosonic string in [16, 9].

In order to obtain new non-trivial solutions we turn to the following property of equations (3.1a): the antisymmetric background field (2.7) depends on X^0 but the corresponding gauge invariant tensor $H_{\lambda\mu\nu}$ does not (moreover, it is constant). Making a gauge transformation (2.5) with

$$\Lambda_\alpha = \tfrac{1}{2} \tilde{\kappa} \epsilon_{0\alpha\beta} X^\beta \qquad \alpha = 0, 1, 2 \tag{3.2}$$

(all other components of Λ vanish) one obtains a static (i.e. X^0-independent) background field which is equivalent to that given by (2.7). The corresponding action is written as

$$\tilde{S} = \kappa \int \mathrm{d}^2\xi \left\{ g^{\frac{1}{2}} + \frac{\lambda}{2}\big(X^2(\partial_0 X^0 \partial_1 X^1 - \partial_0 X^1 \partial_1 X^0) - X^1(\partial_0 X^0 \partial_2 X^2 - \partial_0 X^2 \partial_2 X^0)\big) \right\} \tag{3.3}$$

where

$$\lambda = 3\frac{\tilde{\kappa}}{\kappa}. \tag{3.4}$$

We shall use the action (3.3) instead of (2.11) when discussing the static potential produced by the string.

Due to the invariance of the action (3.2) under world-sheet reparametrizations the solutions to (3.1) are not completely determined by any boundary conditions and gauge fixing is necessary. For obvious reasons the conformal gauge is not convenient in the study of static configurations.

We choose parameters $\xi^0 \equiv \tau$ and $\xi^1 \equiv \eta$ on the world-sheet such that

$$\chi^0\{X; \tau, \eta\} \equiv X^0(\tau, \eta) - \tau = 0. \tag{3.5}$$

Under this condition (which fixes the gauge partially) the action (3.3) is written as

$$\tilde{S} = \kappa \int \mathrm{d}^2\xi \left(g^{\frac{1}{2}} + \frac{\lambda}{2}(X^2 \partial_\eta X^1 - X^1 \partial_\eta X^2) \right) \tag{3.6}$$

where

$$g = \det \begin{pmatrix} 1 + \partial_\tau \boldsymbol{X} \cdot \partial_\tau \boldsymbol{X} & \partial_\tau \boldsymbol{X} \cdot \partial_\eta \boldsymbol{X} \\ \partial_\tau \boldsymbol{X} \cdot \partial_\eta \boldsymbol{X} & \partial_\eta \boldsymbol{X} \cdot \partial_\eta \boldsymbol{X} \end{pmatrix} \tag{3.7}$$

and $\boldsymbol{X} = (X^1, X^2, \ldots, X^{D-1})$. For the static configurations $\{X^0 = \tau, \ \boldsymbol{X} = \boldsymbol{X}(\tau, \eta)\}$ the equations of motion (3.2) read

$$\begin{aligned} &\partial_\eta((\partial_\eta \boldsymbol{X} \cdot \partial_\eta \boldsymbol{X})^{-\frac{1}{2}} \partial_\eta X_1) + \lambda \partial_\eta X_2 = 0 \\ &\partial_\eta((\partial_\eta \boldsymbol{X} \cdot \partial_\eta \boldsymbol{X})^{-\frac{1}{2}} \partial_\eta X_2) - \lambda \partial_\eta X_1 = 0 \\ &\partial_\eta((\partial_\eta \boldsymbol{X} \cdot \partial_\eta \boldsymbol{X})^{-\frac{1}{2}} \partial_\eta X_j) = 0 \qquad j = 3, 4, \ldots, D-1. \end{aligned} \tag{3.8}$$

Equations (3.8) are invariant under reparametrizations $\delta\tau = 0, \ \delta\eta = \varepsilon(\eta)$. A solution can be written as

$$\begin{aligned} X_1^{\mathrm{cl}}(\eta) &= \frac{a}{\lambda}\left(\cos\left(\lambda \int_0^\eta f(s)\,\mathrm{d}s\right) - 1\right) \\ X_2^{\mathrm{cl}}(\eta) &= \frac{a}{\lambda} \sin\left(\lambda \int_0^\eta f(s)\,\mathrm{d}s\right) \\ X_3^{\mathrm{cl}}(\eta) &= b \int_0^\eta f(s)\,\mathrm{d}s \\ X_k^{\mathrm{cl}}(\eta) &= 0 \qquad k = 4, \ldots, D-1 \end{aligned} \tag{3.9}$$

where $0 \leqslant \eta \leqslant \Theta, \ a^2 + b^2 = 1$ and f is a positive function. The solution (3.9) is chosen to satisfy the initial conditions

$$\boldsymbol{X}(0) = 0 \tag{3.10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \boldsymbol{X} = (0, \ af(0), \ bf(0), \ 0, \ldots, \ 0)\,. \tag{3.10b}$$

It is natural to introduce

$$a = \sin\phi \qquad b = \cos\phi \qquad 0 \leqslant \phi \leqslant 2\pi. \tag{3.11}$$

However, for brevity we also keep the notation a, b. Any solution to equations (3.8) can be cast in the form (3.9) by an appropriate space rotation and translation.

A convenient gauge-fixing condition in the vicinity of the solution (3.9) is

$$\chi^1[X; \tau, \eta] \equiv \alpha[X; \tau, \eta] - \eta = 0 \tag{3.12}$$

where

$$\alpha[X; \tau, \eta] = \int_0^\eta \frac{(X_1(\tau, s) + a/\lambda)\, \partial_s X_2(\tau, s) - X_2(\tau, s) \partial_s X_1(\tau, s)}{(X_1(\tau, s) + a/\lambda)^2 + X_2^2(\tau, s)}\,\mathrm{d}s. \tag{3.13}$$

The condition (3.12) is local because α is an integral of a closed 1-form. Imposing this condition on a solution (3.9) (with $a \neq 0$) one finds $f \equiv \lambda^{-1}$ and, consequently,

$$X_1^{\mathrm{cl}}(\eta) = \frac{a}{\lambda}(\cos\eta - 1) \tag{3.14a}$$

$$X_2^{\mathrm{cl}}(\eta) = \frac{a}{\lambda} \sin\eta \tag{3.14b}$$

$$X_3^{\mathrm{cl}}(\eta) = \frac{b}{\lambda}\eta \tag{3.14c}$$

$$X_k^{\mathrm{cl}}(\eta) = 0 \qquad k = 4, \ldots, D-1 \quad 0 \leqslant \eta \leqslant \Theta. \tag{3.14d}$$

We will consider two types of solution: strings with fixed ends and closed strings. A closed string solution is given by (3.14) where $b = 0$ ($a^2 = 1$) and $\Theta = 2\pi N$ (N is the 'winding number'). The energy of a closed string solution is given by

$$E_N^{\mathrm{cl}} = \frac{\pi\kappa N}{\lambda}. \tag{3.15}$$

This quantity is invariant with respect to translations of the solution. The closed string static solution of lowest non-zero energy is that with $N = 1$. We note that there are no topological arguments for the stability of these solutions.

The energy of an open static classical string is written as

$$V^{\mathrm{cl}}(\Theta) = \frac{\kappa}{\lambda}\left(\frac{1+b^2}{2}\Theta + \frac{1-b^2}{2}\sin\Theta\right). \tag{3.16}$$

It is more instructive to use the distance between the ends of string

$$r = \frac{1}{\lambda}\sqrt{4a^2\sin\frac{\Theta}{2} + b^2\Theta^2}, \tag{3.17}$$

and the directional cosine

$$c = \frac{X_3^{\mathrm{cl}}(\Theta)}{r} \tag{3.18}$$

as parameters for the solution instead of Θ and ϕ. By using the expansions

$$\Theta = \lambda r + \frac{1-c^2}{24}\lambda^3 r^3 + \mathrm{O}(\lambda^5 r^5) \tag{3.19a}$$

$$b = c\left(1 - \frac{1-c^2}{24}\lambda^2 r^2 + \mathrm{O}(\lambda^4 r^4)\right) \qquad \lambda r \to 0 \tag{3.19b}$$

we find

$$V^{\mathrm{cl}} = \kappa r\left(1 + \frac{1-c^2}{24}\lambda^2 r^2 + \mathrm{O}(\lambda^3)\right) \qquad \lambda \to 0. \tag{3.20}$$

4. Static potential

Let us consider a 'Wilson loop' C in E^D formed by the curves $\{X_0 = 0,\ \boldsymbol{X} = \boldsymbol{X}^{\mathrm{cl}}(\eta);\ 0 \leqslant \eta \leqslant \theta\}$, $\{X_0 = \tau,\ \boldsymbol{X} = \boldsymbol{X}^{\mathrm{cl}}(\theta);\ 0 \leqslant \tau T\}$, $\{X_0 = T,\ \boldsymbol{X} = \boldsymbol{X}^{\mathrm{cl}}(\eta);\ 0 \leqslant \eta \leqslant \theta\}$, $\{X_0 = \tau,\ \boldsymbol{X} = \boldsymbol{X}^{\mathrm{cl}}(0);\ 0 \leqslant \tau \leqslant T\}$. The average over the surfaces enclosed by the loop [17] is written as

$$\psi(\mathcal{C}) = \int \mathrm{D}\,X\ \delta(\chi^0)\,\delta(\chi^1)\,\Delta\,\mathrm{e}^{-\tilde{S}} \tag{4.1}$$

where the Faddeev–Popov determinant Δ is a constant and will be omitted. One expects the asymptotic behaviour

$$\psi(\mathcal{C}) \sim \mathrm{e}^{-VT} \qquad T \to \infty \tag{4.2}$$

where V can be identified as the ground-state energy of the system (the static potential in the case of an open string which depends on the distance between the ends of the string and its orientation).

We perform the integration on X^0 in (4.1) and make a change† of the integration variables X_1, X_2 (leaving the remaining variables unchanged):

$$X_1(\tau,\eta) = (2Y(\tau,\eta))^{\frac{1}{2}}\cos\varphi(\tau,\eta) - \frac{a}{\lambda}$$
$$X_2(\tau,\eta) = (2Y(\tau,\eta))^{\frac{1}{2}}\sin\varphi(\tau,\eta). \tag{4.3}$$

Using these new variables one replaces (3.14*a*) and (3.14*b*) by

$$Y^{\text{cl}} = \frac{a^2}{2\lambda^2} \tag{4.4a}$$

$$\varphi^{\text{cl}}(\eta) = \eta. \tag{4.4b}$$

The gauge-fixing condition (3.12) is written as

$$\chi^1 \equiv \varphi(\tau,\eta) - \eta = 0 \tag{4.5}$$

and the integration on φ can be easily done yielding

$$\psi(C) = \int \mathcal{D}Y \prod_{k=3}^{D-1} \mathcal{D}X_k\, \mathrm{e}^{-S^{(\text{gf})}} \tag{4.6}$$

where

$$S^{(\text{gf})} = \kappa \int_0^T \mathrm{d}\tau \int_0^\theta \mathrm{d}\eta\, (\sqrt{g} - \lambda Y) \tag{4.7}$$

is the action $\tilde{S}$ in the gauge specified by (3.5), (4.5). Using $\kappa^{-1/2}$ as a loop expansion parameter we make a shift of the integration variables‡:

$$Y = Y^{\text{cl}} + \kappa^{-\frac{1}{2}}\mu^{-\frac{3}{2}}y$$
$$X_3 = X_3^{\text{cl}} + \kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}}z_3 \tag{4.8}$$
$$X_k = \kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}}z_k \qquad k = 4,\ldots,D-1.$$

assuming zero boundary conditions on y, z_k $(k = 3,\ldots,D-1)$.

Expanding $S^{(\text{gf})}$ in powers of κ

$$S^{(\text{gf})} = TV^{\text{cl}}(\Theta) + S^{(2)} + \mathrm{O}(\kappa) \tag{4.9}$$

and changing the variables y, z_3:

$$y = \left(\frac{\mu}{\lambda}a\right)^{1/2} u \qquad z_3 = \left(\frac{\mu}{\lambda}a\right)^{-1/2} v \tag{4.10}$$

we rewrite the bilinear term in a more symmetric form:

$$S^{(2)} = \frac{1}{2\mu^2}\int_0^T \mathrm{d}\tau \int_0^\Theta \mathrm{d}\eta \Bigg((\partial_\tau u)^2 + \lambda^2((\partial_\eta u)^2 - a^2) + (\partial_\tau v)^2 + \lambda^2(\partial_\eta v)^2$$
$$- \frac{\lambda^2 b}{2}(u\partial_\eta v - v\partial_\eta u) + \frac{\mu}{\lambda}\sum_{k=4}^{d-1}((\partial_\eta z_k)^2 + \lambda^2(\partial_\eta z_k)^2)\Bigg). \tag{4.11}$$

† The Jacobians of all the changes of variables which we make are constant.

‡ We prefer to introduce the arbitrary mass scale μ explicitly.

Inserting (4.9) in the functional integral we find, in the semiclassical approximation,

$$\psi(\mathcal{C}) = \exp\left(-TV^{\mathrm{cl}}\right) \int \mathcal{D}u\ \mathcal{D}v \prod_{k=4}^{D-1} \mathcal{D}z_k\ \mathrm{e}^{-S^{(2)}}. \tag{4.12}$$

Taking into account (4.2) one finds that in the expansion of the static potential

$$V = \kappa^{-1}V_{-1} + V_0 + \cdots \tag{4.13}$$

the first term reproduces the classical ground-state energy, while the second one represents the one-loop correction:

$$V_0 = \lim_{T\to\infty} \frac{1}{2T}\left(\mathrm{Tr}\ \log \frac{\widehat{\mathcal{D}}_0 - \widehat{\mathcal{D}}_1}{\mu^2} + (D-4)\ \mathrm{Tr}\ \log\left(-\frac{\partial_\tau^2 + \lambda^2\partial_\eta^2}{\lambda\mu}\right)\right) \tag{4.14}$$

where

$$\widehat{\mathcal{D}}_0 = -\sigma_0\partial_\tau^2 - \lambda^2(\sigma_0\partial_\eta^2 + \tfrac{1}{2}\mathrm{i}b\sigma_2\partial_\eta + \tfrac{1}{2}a^2\sigma_0) \tag{4.15a}$$

$$\widehat{\mathcal{D}}_1 = \frac{\lambda^2 a^2}{2}\sigma_3 \tag{4.15b}$$

σ_k $(k = 1, 2, 3)$ are Pauli matrices and $\sigma_0 = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. The second term in (4.13) has been calculated [18] by means of analytic regularization. The result is

$$\mathrm{Tr}\ \log\left(-\frac{\partial_\tau^2 + \lambda^2\partial_\eta^2}{\lambda\mu}\right) \sim -\frac{\lambda\pi T}{12\Theta} \qquad T\to\infty. \tag{4.16}$$

The leading term does not depend on the (arbitrary) mass scale μ.

The spectrum of the operator† $\widehat{\mathcal{D}} = \widehat{\mathcal{D}}_0 - \widehat{\mathcal{D}}_1$ is unknown, so we assume that

$$\mathrm{Tr}\ \log\frac{\widehat{\mathcal{D}}_0 - \widehat{\mathcal{D}}_1}{\mu^2} = \mathrm{Tr}\ \log\frac{\widehat{\mathcal{D}}_0}{\mu^2} + \mathrm{Tr}\ \log(1 - \widehat{\mathcal{D}}_0^{-1/2}\widehat{\mathcal{D}}_1\widehat{\mathcal{D}}_0^{-1/2}) \tag{4.17}$$

and treat the second term as perturbation. The spectrum of $\widehat{\mathcal{D}}_0$ is positive provided

$$\Theta < \frac{4\pi}{\sqrt{1+7a^2}}. \tag{4.18}$$

Indeed

$$\widehat{\mathcal{D}}_0 = \exp\left(-\frac{\mathrm{i}b\eta}{4}\sigma_2\right)\tilde{\widehat{\mathcal{D}}}_0 \exp\left(\frac{\mathrm{i}b\eta}{4}\sigma_2\right) \tag{4.19}$$

where

$$\tilde{\widehat{\mathcal{D}}}_0 = -\sigma_0\left(\partial_\tau^2 + \lambda^2\left(\partial_\eta^2 + \frac{1+7a^2}{16}\right)\right). \tag{4.20}$$

The spectrum of $\tilde{\widehat{\mathcal{D}}}_0$, and, therefore, that of $\widehat{\mathcal{D}}_0$, is positive if (4.18) is fulfilled. Taking into account (4.18) and (4.19) one finds

$$F_0 = \mathrm{Tr}\ \log\frac{\widehat{\mathcal{D}}_0}{\mu^2} = 2\,\mathrm{Tr}\ \log\left(\frac{-\partial_\tau^2 - \lambda^2\left(\partial_\eta^2 + \frac{1}{16}(1+7a^2)\right)}{\mu^2}\right). \tag{4.21}$$

† Dirichlet boundary conditions are assumed.

It can be calculated by the same method [18], although the result is μ-dependent, in contrast to the 'massless' case (see equation (4.16)). Indeed, it can be supposed that

$$F_0 = -2\frac{\partial}{\partial\beta}\,\mathrm{Tr}\left(-\frac{\partial_\tau^2 + \lambda^2(\partial_\eta^2 + (\pi^2/\Theta^2)\rho^2)}{\mu^2}\right)^{-\beta}\Bigg|_{\beta=0}$$

$$= -\frac{\partial}{\partial\beta}\frac{2}{\Gamma(\beta)}\int_0^\infty \mathrm{d}t\; t^{\beta-1}\,\mathrm{Tr}\,\exp\left(\frac{t}{\mu^2}\partial_\tau^2\right)\mathrm{Tr}\,\exp\left(t\frac{\lambda^2}{\mu^2}\left(\partial_\eta^2 + \frac{\pi^2}{\Theta^2}\rho^2\right)\right)\Bigg|_{\beta=0} \qquad (4.22)$$

where the notation

$$\rho = \frac{\Theta\sqrt{1+7a^2}}{4\pi} \qquad (4.23)$$

is used. The operators ∂_τ^2 and $\partial_\eta^2 + q^2$ in (4.22) are considered on $[0, T]$ and $[0, \Theta]$, respectively. Taking into account the asymptotic

$$\mathrm{Tr}\,\exp\left(\frac{t}{\mu^2}\partial_\tau^2\right) \sim \frac{\mu T}{\sqrt{4\pi t}} \qquad \frac{t}{\mu^2 T^2} \to 0 \qquad (4.24)$$

one obtains an asymptotic expression

$$F_0 \sim -\frac{\mu T}{\sqrt{\pi}}\frac{\partial}{\partial\beta}\frac{\Gamma(\beta-\frac{1}{2})}{\Gamma(\beta)}\left(\frac{\mu\Theta}{\lambda\pi}\right)^{2\beta-1} Z\left(\beta-\frac{1}{2}, -\rho^2\right)\Bigg|_{\beta=0} \qquad (4.25)$$

where $Z(s,q)$ is a (generalized) ζ-function (its definition and properties are summarized in the appendix). By using its expansion in the vicinity of $s = -\frac{1}{2}$ we obtain

$$\mathrm{Tr}\,\log\frac{\widehat{\mathcal{D}}_0}{\mu^2} \sim \frac{2\pi\lambda T}{\Theta}\left(-\frac{1}{12} + \sum_{n=1}^{\infty}\left((n^2-\rho^2)^{1/2} - n + \frac{\rho^2}{2n}\right) - \frac{\rho^2}{2}\log\frac{\tilde{\mu}\Theta}{2\pi\lambda}\right) \qquad T \to \infty \qquad (4.26)$$

where $\tilde{\mu} = \mu\exp(2+\gamma)$, γ being the Euler constant.

We are going to find the asymptotic at large T of the second term

$$F_1 = \mathrm{Tr}\,\log\left(1 - \widehat{\mathcal{D}}_0^{-1/2}\widehat{\mathcal{D}}_1\widehat{\mathcal{D}}_0^{-1/2}\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{Tr}\left(\widehat{\mathcal{D}}_0^{-1}\widehat{\mathcal{D}}_1\right)^n \qquad (4.27)$$

in (4.17). The odd terms in the expansion (4.27) vanish due to the properties of the Pauli matrices and one finds:

$$F_1 = \tfrac{1}{2}\,\mathrm{Tr}\,\log\left(1 - Q^{-1}\right) \qquad (4.28)$$

where the operator

$$Q = \widehat{\mathcal{D}}_1^{-1}\widehat{\mathcal{D}}_0\widehat{\mathcal{D}}_1^{-1}\widehat{\mathcal{D}}_0 = \frac{4\sigma_0}{\lambda^4 a^4}\left(\left(\partial_\tau^2 + \lambda^2\left(\partial_\eta^2 + \frac{a^2}{2}\right)\right)^2 + \frac{\lambda^4 b^2}{4}\partial_\eta^2\right) \qquad (4.29)$$

is self-adjoint. Equation (4.28) can be rewritten as

$$F_1 = \tfrac{1}{2}\,\mathrm{Tr}\,(\log(Q-1) - \log Q) \qquad (4.30)$$

and introducing the resolvent

$$R(z) = (Q - z)^{-1} \qquad (4.31)$$

one obtains

$$F_1 = -\frac{1}{2}\int_0^1 \mathrm{d}z\;\; \mathrm{Tr}\,R(z) \qquad (4.32)$$

providing the function

$$\mathrm{Tr}\, R(z) = 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{4}{\lambda^4 a^4}\left(\left(\left(\frac{\pi m}{T}\right)^2 + \lambda^2\left(\left(\frac{\pi n}{\Theta}\right)^2 - \frac{a^2}{2}\right)\right)^2 - \frac{1}{4}\lambda^4 b^2\left(\frac{\pi n}{\Theta}\right)^2\right) - z\right)^{-1} \tag{4.33}$$

is well defined for $z \in [0, 1]$. A straightforward calculation shows that $R(z)$ is well defined and equation (4.32) can be used if

$$\Theta < \frac{2\pi}{\sqrt{1+3a^2}}. \tag{4.34}$$

We assume that (4.34) is fulfilled (it is easy to check that this condition is stronger than (4.18)). Performing the sum on m in (4.33) we find the following expression for the leading term at large T:

$$\mathrm{Tr}\, R^{\mathrm{as}}(z) = \frac{\lambda a^4 T}{4}\sum_{n=1}^{\infty}\left(a^4 z + b^2\frac{\pi^2 n^2}{\Theta^2}\right)^{-1/2}\left(\left(\frac{\pi^2 n^2}{\Theta^2} - \frac{a^2}{2} - \frac{1}{2}\left(a^4 z + b^2\frac{\pi^2 n^2}{\Theta^2}\right)^{1/2}\right)^{-1/2}\right.$$
$$\left. - \left(\frac{\pi^2 n^2}{\Theta^2} - \frac{a^2}{2} + \frac{1}{2}\left(a^4 z + b^2\frac{\pi^2 n^2}{\Theta^2}\right)^{1/2}\right)^{-1/2}\right). \tag{4.35}$$

Then we insert (4.35) in (4.32) and perform the integration. The result is

$$F_1 \sim \frac{\pi\lambda T}{\Theta}\sum_{n=1}^{\infty}\left(\left(n^2 - \left(\frac{b^2\Theta^2}{4\pi^2}n^2 + \frac{a^4\Theta^4}{4\pi^4}\right)^{1/2} - \frac{a^2\Theta^2}{2\pi^2}\right)^{1/2} - \left(n^2 - \frac{b\Theta}{2\pi}n - \frac{a^2\Theta^2}{2\pi^2}\right)^{1/2}\right.$$
$$+\left(n^2 + \left(\frac{b^2\Theta^2}{4\pi^2}n^2 + \frac{a^4\Theta^4}{4\pi^4}\right)^{1/2} - \frac{a^2\Theta^2}{2\pi^2}\right)^{1/2}$$
$$\left. - \left(n^2 + \frac{b\Theta}{2\pi}n - \frac{a^2\Theta^2}{2\pi^2}\right)^{1/2}\right) \qquad T \to \infty. \tag{4.36}$$

Inserting (4.26) and (4.36) in (4.17) and then in (4.14) we finally obtain

$$V_0 = -\frac{\lambda\pi(D-2)}{24\Theta} - \frac{(1+7a^2)\lambda\Theta}{32\pi}\log\frac{\tilde{\mu}\Theta}{2\lambda\pi}$$
$$+\frac{\lambda\pi}{2\Theta}\sum_{n=1}^{\infty}\left(2\left(\left(n^2 - \frac{1+7a^2}{16}\frac{\Theta^2}{\pi^2}\right)^{1/2} - n + \frac{1+7a^2}{32n}\frac{\Theta^2}{\pi^2}\right)\right.$$
$$+\left(n^2 - \left(\frac{b^2\Theta^2}{4\pi^2}n^2 + \frac{a^4\Theta^4}{4\pi^4}\right)^{1/2} - \frac{a^2\Theta^2}{2\pi^2}\right)^{1/2} - \left(n^2 - \frac{b\Theta}{2\pi}n - \frac{a^2\Theta^2}{2\pi^2}\right)^{1/2}$$
$$+\left(n^2 + \left(\frac{b^2\Theta^2}{4\pi^2}n^2 + \frac{a^4\Theta^4}{4\pi^4}\right)^{1/2} - \frac{a^2\Theta^2}{2\pi^2}\right)^{1/2}$$
$$\left. - \left(n^2 + \frac{b\Theta}{2\pi}n - \frac{a^2\Theta^2}{2\pi^2}\right)^{1/2}\right). \tag{4.37}$$

This expression has been derived under the assumption that (4.34) is satisfied. If $\Theta > 2\pi(1+3a^2)^{-1/2}$ the static potential acquires a non-zero imaginary part indicating an instability in the string. The only exception is the transversal string ($a = 0$). In that case

$F_1 \equiv 0$ and the stability condition is given by (4.18) instead of (4.34). In the case $b = 0$ a result is reproduced which has been obtained in a previous paper [14] by using a different gauge-fixing condition.

In the limit of a vanishing topological term in the action, i.e. $\lambda \to 0$ in (3.3), this expression simplifies considerably. By inserting (3.19) in (4.36), we find

$$V_0 = -\frac{\pi(D-2)}{24r} + \frac{\pi(D-2)(1-c^2)}{24^2}\lambda^2 r - \frac{8-7c^2}{32\pi}\lambda^2 r \log\frac{\tilde{\mu}r}{2\pi} + \mathrm{o}(\lambda^2) \qquad \lambda \to 0. \tag{4.38}$$

The first term reproduces the well known universal correction [19, 17]. The second could be absorbed by a renormalization of $\tilde{\mu}$. The third term is a *log* correction and, in contrast to the second, does not vanish even for the transversal string ($c^2 = 1$).

5. Closed strings

Trying to evaluate the one-loop contribution to the energy of a closed static string we write down an expression similar to (4.14). Zero boundary conditions have to be replaced by periodic ones (with period $2\pi N$). The operator $\widehat{\mathcal{D}}_0$, equation (4.15a), decouples (because $a = 1,\ b = 0$ for a closed static solution) and for the one-loop correction one obtains

$$\Delta E_N = \lim_{T\to\infty}\frac{1}{2T}\left(\mathrm{Tr}\,\log\left(-\frac{\partial_\tau^2 + \lambda^2(\partial_\eta^2+1)}{\mu^2}\right) + \mathrm{Tr}\,\log\left(-\frac{\partial_\tau^2 + \lambda^2\partial_\eta^2}{\mu^2}\right)\right.$$
$$\left. + (D-4)\,\mathrm{Tr}\,\log\left(-\frac{\partial_\tau^2+\lambda^2\partial_\eta^2}{\lambda\mu}\right)\right). \tag{5.1}$$

However, the operator $-\partial_\eta^2 - 1$ (on $[0,\ 2\pi N]$, with periodic boundary conditions) possesses the eigenvalues

$$\epsilon_n = \frac{n^2}{N^2} - 1 \qquad n = 0, 1, \ldots \tag{5.2}$$

which are negative for $n < N$. Therefore any closed static solution is unstable.

6. Conclusions

We have shown that the action including a topological term [6] admits the same constraint algebra and residual symmetry (to the conformal gauge) as those for the ordinary bosonic string. Here the Virasoro algebra also appears twice; first as a constraint algebra and second as the residual symmetry of the action in conformal gauge. This residual symmetry algebra can admit an extension to W_∞ algebra. The static potential (in the semiclassical approximation) was found for an open string in that specific background. However, the solutions of the equations of motions are changed, hence the ground-state energy also is changed. For example, only waves with definite chirality are admissible. As a consequence the structure of the one-loop correction is rather complicated: it includes a log term in particular. The critical distance depends on the orientation of the string. In the limit of vanishing topological term the correction coincides with the well known universal one [18].

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Appendix

The generalized ζ-function

$$Z(s,q)=\sum_{n=1}^{\infty}(n^2+q)^{-s} \qquad \mathrm{Re}\, s>\tfrac{1}{2} \quad |\arg(1+q)|<\pi \tag{A1}$$

which can also be represented as

$$Z(s,q)=\frac{1}{\Gamma(s)}\int_0^{\infty}\mathrm{d}t\; t^{s-1}\sum_{n=1}^{\infty}\exp(-t(n^2+q)) \qquad \mathrm{Re}\, s>\tfrac{1}{2} \quad \mathrm{Re}(1+q)>0 \tag{A2}$$

admits a continuation in the whole s-plane, except for the points

$$s=\tfrac{1}{2}-k \qquad k=1,2,\ldots \tag{A3}$$

where it has poles. Indeed, by expansion in powers of q,

$$\begin{aligned} Z(s,q) &= \sum_{k=0}^{\infty}\frac{\Gamma(1-s)\zeta(2s+2k)}{k!\Gamma(1-s-k)}q^k \\ &= \sum_{k=0}^{\infty}\frac{\Gamma(k+s)\zeta(2s+2k)}{k!\Gamma(s)}(-q)^k. \end{aligned} \tag{A4}$$

A convenient representation for our purposes is

$$Z(s,q)=\zeta(2s)-sq\zeta(2s+2)+\sum_{n=1}^{\infty}\left(\left(n^2+q\right)^{-s}-n^{-2s}+sqn^{-2s-2}\right)$$
$$\mathrm{Re}\, s>-\tfrac{3}{2} \quad |\arg(1+q)|<\pi. \tag{A5}$$

References

[1] Scherk J and Schwarz J H 1974 *Nucl. Phys.* B **81** 155
[2] Yoneya T 1974 *Prog. Theor. Phys.* **51** 1907
[3] Fradkin E S and Tseytlin A A 1985 *Phys. Lett.* **158B** 316; *Nucl. Phys.* B **261** 1
[4] Callan C G, Friedan D, Martinec E and Perry M J 1985 *Nucl. Phys.* B **262** 593
[5] Ibanez L E and Lust D 1992 *CERN Report* CERN-TH.6380/92
[6] Zaikov R P 1991 *Mod. Phys. Lett.* **16A** 1453
[7] Sotkov G M, Stanishkov M and Zhu Chuan-Jie 1991 *Nucl. Phys.* B **356** 245
[8] Narain K S, Sarmadi M H and Witten E 1987 *Nucl. Phys.* B **279** 369
[9] Shapere A and Wilczek F 1989 *Nucl. Phys.* B **320** 669
Giveon A, Rabinovici E and Veneziano G 1989 *Nucl. Phys.* B **322** 167
[10] Polyakov A M 1981 *Phys. Lett.* **103B** 207
[11] Curtright T L and Zachos C K 1984 *Phys. Rev. Lett.* **53** 1799
[12] Kolb M and Ramond P 1974 *Phys. Rev.* D **9** 2273
[13] Rohm R and Witten E 1986 *Ann. Phys.* **170** 454

[14] Zaikov R P and Zlatev S I 1991 *Phys. Lett.* **269B** 283
[15] Bergshoeff E, Pope C N, Romans L J, Sezgin E, Shen X and Stelle K S 1990 *Phys. Lett.* **243B** 350
[16] Narain K S 1986 *Phys. Lett* **169B** 41
[17] Lüscher M 1981 *Nucl. Phys.* B **180** 317
[18] Alvarez O 1981 *Phys. Rev.* D **24** 440
[19] Lüscher M, Symanzik K and Weisz P 1980 *Nucl. Phys.* B **173** 365